Finite Math - Fall 2018

Lecture Notes - 9/20/2018

Homework

- Section 3.2 - 41, 42, 47, 49, 71ab, 73, 75, 78

Section 3.2 - Compound and Continuous Compound Interest

Continuous Compound Interest. Consider again the formulation of compound interest given by

$$A = P\left(1+\frac{r}{m}\right)^{mt}$$

We can do the following manipulation to this expression

$$\begin{aligned} A &= P\left(1+\frac{r}{m}\right)^{mt} \\ &= P\left(1+\frac{r}{m}\right)^{mt\cdot\frac{r}{r}} \\ &= P\left(1+\frac{r}{m}\right)^{\left(\frac{m}{r}\right)rt} \\ &= P\left(1+\frac{1}{x}\right)^{xrt} \qquad \left(x=\frac{m}{r}\right) \\ &= P\left[\left(1+\frac{1}{x}\right)^{x}\right]^{rt} \end{aligned}$$

Now, if we let the number of compounding periods per year m get very very large, then x also gets very large, and we see that the future value becomes

$$A = Pe^{rt}.$$

Definition 1 (Continuous Compound Interest)**.** *Principal P invested at an annual nominal rate r will have future value*

$$A = Pe^{rt}$$

after time t (in years).

Compounding interest continuously gives the absolute largest amount of interest that can be accumulated in the time period t.

Example 1. *If* $\$1,000$ *is invested at* 6% *interest compounded continuously, what is the value of the investment after* 8 *years? Round answers to the nearest cent.*

Solution. *The principal is* $\$1,000$ *and the interest rate is* $r = 0.06$ *with a time of* $t = 8$ *years, so the future value in this case is*

$$A = \$1,000e^{(0.06)(8)} = \$1,000e^{0.48} = \$1,616.07$$

which is we see is larger than any of the others.

Example 2. *If* $\$2,000$ *is invested at* 7% *compounded (a) annually, (b) quarterly, (c) monthly, (d) daily, (e) continuously, what is the amount after* 5 *years? Round answers to the nearest cent. (Assume* 365 *days in a year.)*

Solution.

(a) \$2805.10

(b) \$2829.56

(c) \$2835.25

(d) \$2838.04

(e) \$2838.14

As before, we can use these compound interest models to figure out how much we should invest now to achieve a desired future value.

We can also look to see how long something will take to mature given the principal, the growth rate, and the desired future value. The power rule for logarithms comes especially in handy here: $\log_b M^p = p \log_b M$.

Example 3. *How long will it take* $\$10,000$ *to grow to* $\$25,000$ *if it is invested at* 8% *compounded quarterly?*

Solution. *Here, we have* $P = \$10,000$, $A = \$25,000$, $r = 0.08$, $i = \frac{0.08}{4} = 0.02$, *thus the model gives us*

$$\$25,000 = \$10,000(1 + 0.02)^n$$

and so we can solve for the number of compounding periods required.

$$\begin{aligned} \$25,000 &= \$10,000(1 + 0.02)^n \\ 2.5 &= 1.02^n \\ \ln 2.5 &= n \ln 1.02 \\ n &= \frac{\ln 2.5}{\ln 1.02} = 46.27 \end{aligned}$$

So, this means we need 47 *quarters to achieve* $\$25,000$*, or* 11 *years and* 3 *quarters.*

Example 4. *How long will it take money to triple if it is invested at (a) 5% compounded daily? (b) 6% compounded continuously?*

Solution.

(a) 8,021 *days (about* 21.975 *years)*

(b) 18.310 *years*

We can also look to figure out the desired interest rate if we know the present value, the length of time, and the desired future value.

Example 5. *The Russell Index tracks the average performance of various groups of stocks. On average, a* \$10,000 *investment in mid-cap growth funds over a* 10-*year period would have grown to* \$63,000. *(A mid-cap fund is a type of stock fund that invests in mid-sized companies. See Investopedia for more information.) What annual nominal rate would produce the same growth if interest were compounded (a) annually, (b) continuously. Express answers as a percentage, rounded to three decimal places.*

Solution. *Here, we have* $P = \$10,000$, $t = 10$, *and* $A = \$63,000$ *as given values. We are looking for the rate* r *in both cases.*

(a) *If interest is compounded annually, then the interest per compounding period is* $i = \frac{r}{1} = r$ *and the number of compounding periods is* $n = 1(10) = 10$. *So the formula gives us*

$$\$63,000 = \$10,000(1+r)^{10}.$$

We first need to isolate $(1+r)^{10}$, *then take the* 10^{th} *root of both sides (recall that* $x^{\frac{1}{10}} = \sqrt[10]{x}$*)*

$$\begin{aligned} 63,000 &= 10,000(1+r)^{10} \\ 6.3 &= (1+r)^{10} \\ \sqrt[10]{6.3} &= 1+r \\ r &= \sqrt[10]{6.3} - 1 = 1.20208 - 1 = 0.20208 = 20.208\% \end{aligned}$$

(b) *If interest is compounded continuously, then*

$$\$63,000 = \$10,000e^{r(10)} = \$10,000e^{10r}$$

we first isolate the exponential term, then use ln *to get the exponent out:*

$$\begin{aligned} 63,000 &= 10,000e^{10r} \\ 6.3 &= e^{10r} \\ \ln 6.3 &= 10r \\ r &= \frac{\ln 6.3}{10} = 0.18405 = 18.405\% \end{aligned}$$

Example 6. *A promissory note will pay* $\$50,000$ *at maturity* 6 *years from now. If you pay* $\$28,000$ *for the note now, what rate would you earn if interest were compounded (a) quarterly, (b) continuously?*

Solution.

(a) 9.78%

(b) 9.66%